

# Automorphisms of Real 4 Dimensional Lie Algebras and the Invariant Characterization of Homogeneous 4-Spaces

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## Abstract

The automorphisms of all 4-dimensional, real Lie Algebras are presented in a comprehensive way. Their action on the space of  $4 \times 4$ , real, symmetric and positive definite, matrices, defines equivalence classes which are used for the invariant characterization of the 4-dimensional homogeneous spaces which possess an invariant basis.

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# 1 Introduction

Automorphisms of 3-dimensional, real Lie Algebras [1], have been proven a powerful tool for analyzing the dynamics of 3+1 Bianchi Cosmological Models [2]. At the classical level, time-dependent automorphisms inducing diffeomorphisms can be used to simplify the line element – and thus the Einstein’s Field Equations – without loss of generality [3]. They also provide an algorithm for counting the number of essential constants; the results obtained agree for all Bianchi Types with the preexisting results [4] but, unlike these, the algorithm can be extended to 4 or more dimensions. At the quantum level, outer automorphisms provide integrals of motion of the classical Hamiltonian dynamics; their quantum analogues can be used to reconcile quantum Hamiltonian dynamics with the kinematics of homogeneous 3-spaces [5].

A corresponding analysis of these issues for the case of 4+1 spatially homogeneous geometries, seems very interesting in itself. It could also prove valuable for the nowadays fashionable brane world models. As a first step in implementing such an analysis, we exhibit the automorphisms for all, real, 4-dimensional Lie Algebras (a first treatise of the subject can be found in [6]) and subsequently use them, to invariantly describe homogeneous 4-spaces.

## 2 Automorphisms

Before exhibiting the results on the automorphisms and their generators, we briefly recall some basic elements of the Theory of Lie Groups. Topological issues will not concern us, since at this stage of study, are rather irrelevant.

Let  $V_N$ , be a vector space over the field  $\mathbb{R}$ . For each point  $x^m$  in the space, a set of transformations  $\tilde{x}^m = f^m(x^n; \alpha^\mu)$  depending on some parameters  $\alpha^\mu$  (with Greek indices ranging on the closed interval  $[1, \dots, M]$ , while Latin ones, on the closed  $[1, \dots, N]$ ) is defined, endowed with the following properties:

- The parameters are essential, i.e. they are not functions of others; rather, they take values on a compact domain.
- There are particular values for each and every parameter  $\alpha^\mu$  – which without loss of generality can be taken to be all zero – such that  $x^m = f^m(x^n; 0, \dots, 0)$ . In other words, the identity transformation, is reached continuously, when all the parameters reach this particular set of values (here the zeros).
- The Jacobian of the transformation,  $J_n^m = |\partial f^m(x^k; \alpha^\mu) / \partial x^n|$ , is non vanishing on its entire domain of definition. For, every transformation, at least locally, must be invertible.

If one assumes that the parameters are small, and expands in Taylor series the transformations (the functions  $f^m$  are taken to be  $C^n$  differentiable, with  $n$  depending on the

application) he will get:

$$\tilde{x}^m = f^m(x^k; 0, \dots, 0) + \alpha^\mu \left( \frac{\partial f^m(x^k; \alpha^\nu)}{\partial \alpha^\mu} \Big|_{\alpha^\nu=0} \right) + \mathcal{O}(\alpha^\mu \alpha^\nu) \quad (2.1)$$

(the Einstein Summation Convention, is in use). Then, a set of  $M$ ,  $N$ -dimensional, vector fields (each for every essential parameter), is associated to the previous infinitesimal transformations:

$$X_\mu^m = \frac{\partial f^m(x^n; \alpha^\nu)}{\partial \alpha^\mu} \Big|_{\alpha^\nu=0} \quad (2.2)$$

These vector fields are called “Generators” and form an Algebra:

$$[X_\mu, X_\nu] = C_{\mu\nu}^\kappa X_\kappa \quad (2.3)$$

which is called Lie Algebra – due to the above mentioned properties. If the quantities  $C_{\mu\nu}^\kappa$  do not depend on the space point  $x^m$ , they are called “Structure Constants”, otherwise “Structure Functions” and the corresponding algebras, open Lie Algebras. In what follows, the Lie Algebras, are assumed to be closed. In this case, the vector space  $V_N$ , admits a Group of Motions (i.e. transformations)  $G_M$  to which the Lie Algebra of the generators of the transformations, is associated. Then it can be proven that  $M < N(N+1)/2$  – see [7], for a detailed analysis.

The Jacobi Identities for the generators, hold:

$$[[X_\mu, X_\nu], X_\kappa] + [[X_\nu, X_\kappa], X_\mu] + [[X_\kappa, X_\mu], X_\nu] = 0 \quad (2.4)$$

or in terms of the structure constants:

$$C_{\mu\nu}^\rho C_{\rho\kappa}^\sigma + C_{\nu\kappa}^\rho C_{\rho\mu}^\sigma + C_{\kappa\mu}^\rho C_{\rho\nu}^\sigma = 0 \quad (2.5)$$

If one contracts the index  $\sigma$  with a contravariant index – e.g.  $\kappa$  –, one gets the Contracted Jacobi Identities:

$$C_{\mu\nu}^\rho C_{\rho\sigma}^\sigma = 0 \quad (2.6)$$

and thus a “natural” quantity emerges, namely  $C_{\rho\sigma}^\sigma \doteq \nu_\rho$ , which is a covector under the action of  $GL(M, \mathbb{R})$  – see [8] and the references therein.

Under a linear mixing, i.e. the action of the  $GL(M, \mathbb{R})$ , of the generators:

$$X_\nu \rightarrow \tilde{X}_\nu = L_\nu^\mu X_\mu \quad (2.7)$$

the structure constants, transform – according to (2.3) – as:

$$C_{\mu\nu}^\kappa \rightarrow \tilde{C}_{\mu\nu}^\kappa = L_\mu^\alpha L_\nu^\beta (L^{-1})_\rho^\kappa C_{\alpha\beta}^\rho \quad (2.8)$$

while:

$$\nu_\nu \rightarrow \tilde{\nu}_\nu = L_\nu^\mu \nu_\mu \quad (2.9)$$

The subset of those transformations (i.e. of the form (2.7)) with respect to which the structure constants are invariant, is the Automorphism Group of the Lie Algebra  $Aut(G)$ . If  $\Lambda_\nu^\mu$  are the matrices of this group, then:

$$C_{\mu\nu}^\kappa = \Lambda_\mu^\alpha \Lambda_\nu^\beta (\Lambda^{-1})_\rho^\kappa C_{\alpha\beta}^\rho \quad (2.10)$$

At first sight, in order to find the Automorphism Group of a given Lie Algebra (i.e. for a given set of non vanishing structure constants), one has to solve the cubic system (2.10), which can be transformed to quadratic by noting that the matrices of interest, are non singular and thus:

$$C_{\mu\nu}^\kappa \Lambda_\kappa^\rho = \Lambda_\mu^\alpha \Lambda_\nu^\beta C_{\alpha\beta}^\rho \quad (2.11)$$

but still, the quest for the solutions, remains a difficult task.

A first simplification may be achieved by observing that:

$$\nu_\nu = \Lambda_\nu^\mu \nu_\mu \quad (2.12)$$

A second simplification, makes use of the Killing-Cartan metric – provided by a famous theorem due to Cartan:

$$g_{\mu\nu} = C_{\beta\mu}^\alpha C_{\alpha\nu}^\beta \quad (2.13)$$

Automorphisms preserve its form i.e. are isometries of this metric:

$$g_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta g_{\alpha\beta} \quad (2.14)$$

The combined use of (2.11), (2.12) and (2.14) makes the first simpler to solve, providing us with useful necessary conditions restricting the  $\Lambda_\nu^\mu$ 's.

In order to find the generators of the Automorphism Group of a given Lie Algebra, it is necessary to consider a family of automorphic matrices which is connected to the identity i.e.  $\Lambda_\nu^\mu = \Lambda_\nu^\mu(\tau)$  for some parameter  $\tau$  such that  $\Lambda_\nu^\mu(0) = I_M$ , with  $I_M$  the  $M$  dimensional identity matrix. Then if one substitutes this family to (2.11), differentiates with respect to this parameter, and sets at zero, one will get:

$$\lambda_\rho^\kappa C_{\mu\nu}^\rho = \lambda_\mu^\rho C_{\rho\nu}^\kappa + \lambda_\nu^\rho C_{\mu\rho}^\kappa \quad (2.15)$$

where:

$$\lambda_\nu^\mu = \left. \frac{d\Lambda_\nu^\mu(\tau)}{d\tau} \right|_{\tau=0} \quad (2.16)$$

is the requested generator. The system (2.15) is linear and thus easy to solve, when the values of the structure constants are given. The number of independent solutions to it, determines the number of the independent parameters of the generators of the Automorphism Group.

The situation in the literature, concerning the 4-dim, real Lie Algebras, is characterized by a certain degree of diversity. The main reason is that, unlike the case of 3-dim, real Lie Algebras, a unique decomposition of the structure constants' tensor in terms of lower rank objects, has not been found. As a result, the presentations of Petrov [7], MacCallum [8] and Patera et. al. [9], differ substantially, especially as far as the number of different real Lie Algebras, is concerned.

In the following the non vanishing structure constants for the various 4-dimensional, real, Lie Algebras (according to Patera et. al. Ref. [9] which is considered to be the most complete and extensive), the automorphism matrices and their generators, are given in Table 1. Also, for each algebra, an irreducible form of a generic,  $4 \times 4$ , symmetric, positive definite, real matrix is given, in Table 2, along with a suggested basis of invariants.

We now come to the invariant description of a homogeneous 4-space. Let  $\sigma_i^\alpha(x)$  denote the basis of one forms, invariant under the action of the symmetry group of motions, acting simply transitively on the space. Then:

$$\sigma_{i,j}^\alpha(x) - \sigma_{j,i}^\alpha(x) = 2C_{\mu\nu}^\alpha \sigma_j^\mu(x) \sigma_i^\nu(x) \quad (2.17)$$

where  $C_{\mu\nu}^\alpha$ , are the structure constants of the corresponding Lie Algebra. Using this basis we can write, in these coordinates the most general, manifestly invariant, line element as:

$$ds^2 = \gamma_{\alpha\beta} \sigma_i^\alpha(x) \sigma_j^\beta(x) dx^i dx^j \quad (2.18)$$

where  $\gamma_{\alpha\beta}$  is a numerical,  $4 \times 4$ , real, positive definite, symmetric matrix. If we consider the class of general co-ordinate transformations (GCT's)  $x^i = f^i(y^m)$  which leave the given basis one forms quasi-form invariant, i.e. those satisfying:

$$\sigma_i^\alpha(x) \frac{\partial x^i}{\partial y^m} = \Lambda_\mu^\alpha \sigma_m^\mu(y) \quad (2.19)$$

then we have a well defined, non trivial action, on the configuration space, spanned by  $\gamma_{\alpha\beta}$ 's, given by:

$$\tilde{\gamma}_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta \gamma_{\alpha\beta} \quad (2.20)$$

The relevant result for 3-spaces, are given in [5] and the generalization to 4-spaces, is obvious. The requirement for  $\Lambda_\beta^\alpha$  to be constant leads, through the integrability conditions for (2.19), to the restrictions:

$$C_{\mu\nu}^\rho \Lambda_\rho^\alpha = \Lambda_\mu^\kappa \Lambda_\nu^\sigma C_{\kappa\sigma}^\alpha \quad (2.21)$$

which reveal  $\Lambda_\beta^\alpha$ , as an element of the automorphism group of the corresponding Lie Algebra. Thus, the configuration space, is divided into equivalence classes by the action of the automorphism group according to (2.20).

In order to have an infinitesimal description of this action, we need to consider the generators  $\lambda_\beta^\alpha$  of  $\Lambda_\beta^\alpha$ . Their defining relations are (2.15). We can easily see that the linear vector fields in the configuration space:

$$X_{(i)} = \lambda_{\mu(i)}^\alpha \gamma_{\alpha\nu} \frac{\partial}{\partial \gamma_{\mu\nu}} \quad (2.22)$$

induce, through their integral curves, exactly the motions (2.20) –  $(i)$  is a collective index corresponding to a choice of base for  $\lambda_\mu^\alpha$  and counts the number of independent vector fields. Thus, if we wish for a scalar function  $\Psi = \Psi(\gamma_{\alpha\beta})$  to change only when we move from one class to another, then we must demand:

$$X_{(i)}\Psi = 0, \quad \forall \quad i \in [1, \dots, d], \quad d < 10 \quad (2.23)$$

The solutions to this system of equations, say  $q^A = q^A(C_{\mu\nu}^\alpha, \gamma_{\mu\nu})$  lead to the finite description of the action of automorphisms – with  $A$  taking its values on the interval  $[1, \dots, 10 - d]$ . By construction, they satisfy:

$$q^{(1)A} = q^{(2)A} \quad \text{for every} \quad (\gamma_{\alpha\beta}^{(1)}, \gamma_{\alpha\beta}^{(2)}) \quad (2.24)$$

connected through an automorphism, as in (2.20). A kind of inverse to this proposition, which completes the finite description of the 4-spaces, discussed here, is contained in the following:

**Theorem 1.** *Let  $\gamma_{\alpha\beta}^{(1)}, \gamma_{\alpha\beta}^{(2)}$  belong to the configuration space. If  $q^{(1)A} = q^{(2)A} \quad \forall \quad A$ , then there is  $\Lambda_\beta^\alpha \in \text{Aut}(G)$  such that  $\gamma_{\mu\nu}^{(2)} = \Lambda_\mu^\alpha \Lambda_\nu^\beta \gamma_{\alpha\beta}^{(1)}$ .*

*Proof.* We first observe that, as seen in Table 2, for each and every Lie Algebra, the – connected to the identity – component of the automorphism group, suffices to bring the generic, positive definite, real,  $4 \times 4$ ,  $\gamma_{\alpha\beta}$  to an irreducible (though not unique) form  $\gamma_{\alpha\beta}^{\text{Ir.}}$ , possessing a number of remaining arbitrary components which equals the number of independent  $q^{(A)}$ 's. Thus, in this ‘gauge’ the hypothesis of the theorem i.e.  $q^{(1)A} = q^{(2)A} \quad \forall \quad A$ , implies – through the implicit function theorem [10] – that  $\gamma_{\alpha\beta}^{\text{Ir. (2)}} = \gamma_{\alpha\beta}^{\text{Ir. (1)}}$ . Therefore, if  $\Lambda_\beta^{(2)\alpha}, \Lambda_\beta^{(1)\alpha}$  are the simplifying automorphisms, i.e.  $\gamma_{\mu\nu}^{(2)} = \Lambda_\mu^{(2)\alpha} \Lambda_\nu^{(2)\beta} \gamma_{\alpha\beta}^{\text{Ir. (2)}}$  and  $\gamma_{\mu\nu}^{(1)} = \Lambda_\mu^{(1)\alpha} \Lambda_\nu^{(1)\beta} \gamma_{\alpha\beta}^{\text{Ir. (1)}}$  then the transformation  $\Lambda_\beta^\alpha = (\Lambda^{-1})_\rho^{(1)\alpha} \Lambda_\beta^{(2)\rho}$  connects  $\gamma_{\mu\nu}^{(2)}$  to  $\gamma_{\mu\nu}^{(1)}$  and obviously belongs to  $\text{Aut}(G)$ . q.e.d.

Returning to the form of the solutions to equations (2.23) it is straightforward to check that every scalar combination of  $C_{\mu\nu}^\lambda$  and  $\gamma_{\mu\nu}, \gamma^{\alpha\beta}$ , is annihilated by all  $X_{(i)}$ 's. The number of independent such scalar contractions is, at most, six: the 10  $\gamma_{\mu\nu}$ 's plus the 12  $C_{\mu\nu}^\lambda$ 's (24 initially independent - 12 independent Jacobi Identities) minus 16 arbitrary elements of  $GL(4, \mathbb{R})$ . The same number is obtained by observing that the automorphism

group always contains the inner automorphism subgroup which has 4 generators thus, there will be at most,  $10-4=6$  independent scalar combinations.

A common, though not unique, suitable basis in the space of all such scalars, valid for all 4-dim, Real Lie Algebras is:

$$q^1, q^2, q^3, q^4, q^5, q^6 \quad (2.25a)$$

where:

$$q^1 = \Pi_{\alpha\beta\mu\nu} \gamma^{\alpha\mu} \gamma^{\beta\nu} \quad (2.25b)$$

$$q^2 = C_{\beta\kappa}^{\alpha} C_{\alpha\lambda}^{\beta} \gamma^{\kappa\lambda} \quad (2.25c)$$

$$q^3 = \Pi_{\alpha\beta\mu\nu} \Pi^{\alpha\beta\mu\nu} \quad (2.25d)$$

$$q^4 = \Pi_{\alpha\beta\kappa\lambda} \Pi_{\mu\nu\rho\sigma} \Pi^{\alpha\beta\mu\rho} \gamma^{\kappa\nu} \gamma^{\lambda\sigma} \quad (2.25e)$$

$$q^5 = \Upsilon_{\alpha} \Upsilon_{\beta} \gamma^{\alpha\beta} \quad (2.25f)$$

$$q^6 = \Omega_{\alpha} \Omega_{\beta} \gamma^{\alpha\beta} \quad (2.25g)$$

with the allocations:

$$\Pi_{\alpha\beta\mu\nu} = C_{\alpha\beta}^{\rho} C_{\mu\nu}^{\sigma} \gamma_{\rho\sigma} \quad (2.26)$$

$$\Upsilon_{\alpha} = \Pi_{\alpha\beta\mu\nu} C_{\kappa\lambda}^{\nu} \gamma^{\beta\lambda} \gamma^{\mu\kappa} \quad (2.27)$$

$$\Omega_{\alpha} = \Pi_{\alpha\beta\mu\nu} C_{\kappa\lambda}^{\nu} \Pi^{\beta\lambda\mu\kappa} \quad (2.28)$$

(Greek indices are raised and lowered with  $\gamma^{\alpha\beta}$  and  $\gamma_{\alpha\beta}$  respectively).

The number of functionally independent  $q^A$ 's is 6, only for a 5 out of the 30 homogeneous 4-spaces, here considered – a fact that is reminiscent of the analogous situation in Bianchi Types, where only Type *VIII* and *IX* possess three functionally independent  $q$ 's. For the rest of the cases, the number of functionally independent  $q^A$ 's is less than 6.

These  $q^A$ 's can serve as ‘coordinates’ of the reduced configuration space on which the Wheeler-DeWitt equation, is to be founded.

### 3 Discussion

We have investigated the action of the automorphism group of all 4-dim, Real Lie Algebras, in the space of  $4 \times 4$ , real, symmetric, and positive definite matrices, which is the configuration space of homogeneous 4-spaces (admitting an invariant basis of one-forms). These automorphisms naturally emerge as the non trivial action of the Diffeomorphism Group on this space. The finite invariant description of these 4-spaces, is given in terms of the scalar combinations of the structure constants with the scale factor matrix. The number of independent such scalars is found to be at most 6. These scalars can be considered either as the independent solutions to (2.23) or as the differential scalar contractions constructed out of the metric and its derivatives.

In 3-dim these are the curvature and/or higher derivative curvature scalars; in 4-dim

there are also scalar combinations of the metric and its derivatives of degree greater than 3 (metric invariants [11]), which can not be expressed as higher derivative curvature invariants.

The  $q^A$ 's given in (2.25b) may be curvature, higher derivative curvature, or metric invariants. They irreducibly characterize the corresponding homogeneous space. These quantities are useful both at the classical and quantum level: classically the irreducible of the scale factor matrix, (which in the light of the Theorem are essentially the  $q^A$ 's) can be used as a starting point for solving the Einstein Field Equations of the corresponding 5-dim cosmological models. Quantum mechanically the Wheeler-DeWitt equation, when an action principle exists, is to be constructed on the reduced configuration space spanned by the  $q^A$ 's. The reasoning of this is the desire to have “gauge” invariant wave functions.

Our analysis does not treat the Kantowski-Sachs like geometries (homogeneous spaces with multiply transitive groups of motions). These are 6 in number [12].

Finally, we would like to mention a word about the computation of the basis of invariants: it can be carried out using a symbolic algebra package (such as Mathematica) and the hard thing is to find a basis valid for all the 30 homogeneous spaces. The price one pays is that one has to consider as many as 10 powers of the structure constants (i.e. 10 derivatives of the metric). This, however, is a worthwhile sacrifice of simplicity, since it will enable one to make comparative studies of the corresponding 5-dimensional Quantum Cosmologies.



## Table Captions

### Table1

The first column gives the names of the Lie Algebras according to [9]. The second, gives the non vanishing structure constants. Column three, exhibits the corresponding automorphism group  $\Lambda_\beta^\alpha$  –along with the disconnected to the identity component, when it exists. Finally, in column four, the generators of the automorphism group  $\lambda_\beta^\alpha$ , are presented.

### Table2

The first column gives the names of the Lie Algebras according to [9]. The second, gives a possible irreducible form for a generic  $4 \times 4$ , real, symmetric and positive definite matrix,  $\gamma_{\alpha\beta}$  as resulting by the action of the corresponding automorphism group. Care has been taken, so that the exhibited reduced form can always be achieved. Finally, in column three, a set of functionally independent metric invariants, is given. Functional independence has been tested using the reduced form of  $\gamma_{\alpha\beta}$ 's –given in column 2. However, since the reduction has been achieved by the appropriate automorphism group, and since this action is nothing but the effect of a general co-ordinate transformation, one concludes that the given metric invariants are functionally independent, with respect to the generic  $\gamma_{\alpha\beta}$ , as well.

TABLE 1

Lie Algebra	Non Vanishing Structure Constants	Automorphisms $\Lambda_\beta^\alpha$	Generators $\lambda_\beta^\alpha$
$4A_1$		$GL(4, \mathfrak{R})$	$GL(4, \mathfrak{R})$
$A_2 \oplus A_1$	$C_{12}^2 = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ a_9 & 0 & a_{11} & a_{12} \\ a_{13} & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ g_5 & g_6 & 0 & 0 \\ g_9 & 0 & g_{11} & g_{12} \\ g_{13} & 0 & g_{15} & g_{16} \end{pmatrix}$
$2A_2$	$C_{12}^2 = 1 \ C_{34}^4 = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_7 & a_8 \\ 1 & 0 & 0 & 0 \\ a_{13} & a_{14} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ g_5 & g_6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,1} \oplus A_1$	$C_{23}^1 = 1$	$\begin{pmatrix} a_{11}a_6 - a_{10}a_7 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & 0 \\ 0 & a_{10} & a_{11} & 0 \\ 0 & a_{14} & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_6 + g_{11} & g_2 & g_3 & g_4 \\ 0 & g_6 & g_7 & 0 \\ 0 & g_{10} & g_{11} & 0 \\ 0 & g_{14} & g_{15} & g_{16} \end{pmatrix}$
$A_{3,2} \oplus A_1$	$C_{13}^1 = 1 \ C_{23}^1 = 1 \ C_{23}^2 = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ 0 & g_1 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,3} \oplus A_1$	$C_{13}^1 = 1 \ C_{23}^2 = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_5 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ g_5 & g_6 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,4} \oplus A_1$	$C_{13}^1 = 1 \ C_{23}^2 = -1$	$\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$ or $\begin{pmatrix} 0 & a_2 & a_3 & 0 \\ a_5 & 0 & a_7 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & g_3 & 0 \\ 0 & g_6 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$

TABLE 1 (Continued)

Lie Algebra	Non Vanishing Structure Constants	Automorphisms $\Lambda_\beta^\alpha$	Generators $\lambda_\beta^\alpha$
$A_{3,5}^\alpha \oplus A_1$ $0 <  \alpha  < 1$	$C_{13}^1 = 1 \ C_{23}^2 = \alpha$	$\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & g_3 & 0 \\ 0 & g_6 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,6} \oplus A_1$	$C_{13}^2 = -1 \ C_{23}^1 = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & -a_1 & a_7 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ -g_2 & g_1 & g_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,7}^\alpha \oplus A_1$ $0 < \alpha$	$C_{13}^1 = \alpha \ C_{13}^2 = -1 \ C_{23}^1 = 1$ $C_{23}^2 = \alpha$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ -g_2 & g_1 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,8} \oplus A_1$	$C_{23}^1 = 1 \ C_{13}^2 = -1 \ C_{12}^3 = -1$	see Appendix	
$A_{3,9} \oplus A_1$	$C_{12}^3 = 1 \ C_{23}^1 = 1 \ C_{31}^2 = 1$	see Appendix	
$A_{4,1}$	$C_{24}^1 = 1 \ C_{34}^2 = 1$	$\begin{pmatrix} a_{11}a_{16}^2 & a_7a_{16} & a_3 & a_4 \\ 0 & a_{11}a_{16} & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_{11} + 2g_{16} & g_7 & g_3 & g_4 \\ 0 & g_{11} + g_{16} & g_7 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & g_{16} \end{pmatrix}$

TABLE 1 (Continued)

Lie Algebra	Non Vanishing Structure Constants	Automorphisms $\Lambda_\beta^\alpha$	Generators $\lambda_\beta^\alpha$
$A_{4,2}^\alpha$  $\alpha \neq (0, 1)$	$C_{14}^1 = \alpha \ C_{24}^2 = 1 \ C_{34}^2 = 1$  $C_{34}^3 = 1$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,2}^1$	$C_{14}^1 = 1 \ C_{24}^2 = 1 \ C_{34}^2 = 1$  $C_{34}^3 = 1$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ a_5 & a_6 & 0 & a_8 \\ 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ g_5 & g_{11} & 0 & g_8 \\ 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,3}$	$C_{14}^1 = 1 \ C_{34}^2 = 1$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_{11} & g_7 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,4}$	$C_{14}^1 = 1 \ C_{24}^1 = 1 \ C_{24}^2 = 1$  $C_{34}^2 = 1 \ C_{34}^3 = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_8 \\ 0 & 0 & a_1 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & g_7 & g_3 & g_4 \\ 0 & g_1 & g_7 & g_8 \\ 0 & 0 & g_1 & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,5}^{\alpha,\beta}$  $-1 \leq \alpha < \beta < 1, \alpha\beta \neq 0$	$C_{14}^1 = 1 \ C_{24}^2 = \alpha \ C_{34}^3 = \beta$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

TABLE 1 (Continued)

Lie Algebra	Non Vanishing Structure Constants	Automorphisms $\Lambda_\beta^\alpha$	Generators $\lambda_\beta^\alpha$
$A_{4,5}^{\alpha,\alpha}$ $-1 \leq \alpha < 1, \alpha \neq 0$	$C_{14}^1 = 1 \ C_{24}^2 = \alpha \ C_{34}^3 = \alpha$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_6 & g_7 & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,5}^{\alpha,1}$ $-1 \leq \alpha < 1, \alpha \neq 0$	$C_{14}^1 = 1 \ C_{24}^2 = \alpha \ C_{34}^3 = 1$	$\begin{pmatrix} a_1 & 0 & a_3 & a_4 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & g_3 & g_4 \\ 0 & g_6 & 0 & g_8 \\ g_9 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,5}^{1,1}$ $C_{14}^1 = 1 \ C_{24}^2 = 1 \ C_{34}^3 = 1$		$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ g_5 & g_6 & g_7 & g_8 \\ g_9 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,6}^{\alpha,\beta}$ $\alpha \neq 0, \beta \geq 0$	$C_{14}^1 = \alpha \ C_{24}^2 = \beta \ C_{24}^3 = -1$ $C_{34}^2 = 1 \ C_{34}^3 = \beta$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_{11} & -g_{10} & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,7}$	$C_{14}^1 = 2 \ C_{24}^2 = 1 \ C_{34}^2 = 1$ $C_{34}^3 = 1 \ C_{23}^1 = 1$	$\begin{pmatrix} a_6^2 & -a_{12}a_6 & -a_{12}(a_6 + a_7) + a_6a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2g_{11} & -g_{12} & -g_{12} + g_8 & g_4 \\ 0 & g_{11} & g_7 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

TABLE 1 (Continued)

Lie Algebra	Non Vanishing Structure Constants	Automorphisms $\Lambda_\beta^\alpha$	Generators $\lambda_\beta^\alpha$
$A_{4,8}$	$C_{23}^1 = 1 \ C_{24}^2 = 1 \ C_{34}^3 = -1$	$\begin{pmatrix} a_{11}a_6 & a_{12}a_6 & a_{11}a_8 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -a_{10}a_7 & -a_{10}a_8 & -a_{12}a_7 & a_4 \\ 0 & 0 & a_7 & a_8 \\ 0 & a_{10} & 0 & a_{12} \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & g_{12} & g_8 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,9}^\beta$ $0 <  \beta  < 1$	$C_{23}^1 = 1 \ C_{14}^1 = 1 + \beta \ C_{24}^2 = 1$ $C_{34}^3 = \beta$	$\begin{pmatrix} a_{11}a_6 & -a_{12}a_6/\beta & a_8a_{11} & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & -g_{12}/\beta & g_8 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,9}^1$	$C_{23}^1 = 1 \ C_{14}^1 = 2 \ C_{24}^2 = 1$ $C_{34}^3 = 1$	$\begin{pmatrix} a_{11}a_6 - a_{10}a_7 & -a_{12}a_6 + a_{10}a_8 & a_8a_{11} - a_7a_{12} & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & -g_{12} & g_8 & g_4 \\ 0 & g_6 & g_7 & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,9}^0$	$C_{23}^1 = 1 \ C_{14}^1 = 1 \ C_{24}^2 = 1$	$\begin{pmatrix} a_{11}a_6 & a_2 & a_8a_{11} & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & g_2 & g_8 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

TABLE 1 (Continued)

Lie Algebra	Non Vanishing Structure Constants	Automorphisms $\Lambda_\beta^\alpha$	Generators $\lambda_\beta^\alpha$
$A_{4,10}$	$C_{23}^1 = 1 \ C_{24}^3 = -1 \ C_{34}^2 = 1$	$\begin{pmatrix} a_6^2 + a_7^2 & a_{12}a_7 - a_6a_8 & -a_{12}a_6 - a_7a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -a_6^2 - a_7^2 & a_{12}a_7 + a_6a_8 & -a_{12}a_6 + a_7a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_7 & -a_6 & a_{12} \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 2g_{11} & -g_8 & -g_{12} & g_4 \\ 0 & g_{11} & -g_{10} & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,11}^\alpha$ $\alpha > 0$	$C_{23}^1 = 1 \ C_{14}^1 = 2\alpha \ C_{24}^2 = \alpha$ $C_{24}^3 = -1 \ C_{34}^2 = 1 \ C_{34}^3 = \alpha$	$\begin{pmatrix} a_6^2 + a_7^2 & -(w1)/(1 + \alpha^2) & -(w2)/(1 + \alpha^2) & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $w1 = a_6(\alpha a_{12} + a_8) + a_7(\alpha a_8 - a_{12})$ $w2 = a_6(a_{12} - \alpha a_8) + a_7(\alpha a_{12} + a_8)$	$\begin{pmatrix} 2g_{11} & g_2 & g_3 & g_4 \\ 0 & g_{11} & -g_{10} & -g_2 + \alpha g_3 \\ 0 & g_{10} & g_{11} & -\alpha g_2 - g_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,12}$	$C_{13}^1 = 1 \ C_{23}^2 = 1 \ C_{14}^2 = -1$ $C_{24}^1 = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & -a_1 & -a_4 & a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} g_6 & -g_5 & -g_8 & g_7 \\ g_5 & g_6 & g_7 & g_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

## A Appendix

We give the automorphisms and their generators for  $A_{3,8} \oplus A_1$  and  $A_{3,9} \oplus A_1$  Lie Algebras, in symbolic (matrix) notation.

$$A_{3,8} \oplus A_1$$

$$\Lambda = \text{Rotation}_{xy} \text{Boost}_{xz} \text{Boost}_{yz} C \quad (\text{A.1})$$

where:

$$\text{Rotation}_{xy} = \begin{pmatrix} \cos(a_1) & \sin(a_1) & 0 & 0 \\ -\sin(a_1) & \cos(a_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.2})$$

$$\text{Boost}_{xz} = \begin{pmatrix} \cosh(a_2) & 0 & \sinh(a_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(a_2) & 0 & \cosh(a_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.3})$$

$$\text{Boost}_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(a_3) & \sinh(a_3) & 0 \\ 0 & \sinh(a_3) & \cosh(a_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.4})$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \quad (\text{A.5})$$

The generator:

$$\begin{pmatrix} 0 & g_2 & g_3 & 0 \\ -g_2 & 0 & g_7 & 0 \\ g_3 & g_7 & 0 & 0 \\ 0 & 0 & 0 & g_{16} \end{pmatrix} \quad (\text{A.6})$$



$$A_{3,9} \oplus A_1$$

$$\Lambda = \text{Rotation}_{xy} \text{Rotation}_{xz} \text{Rotation}_{yz} C \quad (\text{A.7})$$

where:

$$\text{Rotation}_{xy} = \begin{pmatrix} \cos(a_1) & \sin(a_1) & 0 & 0 \\ -\sin(a_1) & \cos(a_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.8})$$

$$\text{Rotation}_{xz} = \begin{pmatrix} \cos(a_2) & 0 & -\sin(a_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(a_2) & 0 & \cos(a_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.9})$$

$$\text{Rotation}_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(a_3) & \sin(a_3) & 0 \\ 0 & -\sin(a_3) & \cos(a_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.10})$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \quad (\text{A.11})$$

The generator:

$$\begin{pmatrix} 0 & g_2 & g_3 & 0 \\ -g_2 & 0 & g_7 & 0 \\ -g_3 & -g_7 & 0 & 0 \\ 0 & 0 & 0 & g_{16} \end{pmatrix} \quad (\text{A.12})$$

TABLE 2

Lie Algebra	(Possible) Reducible Form of $\gamma_{\alpha\beta}$	Functionally Independent Invariants
$4A_1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	None
$A_2 \oplus 2A_1$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma_{24} \\ 0 & 0 & 1 & 0 \\ 0 & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2$
$2A_2$	$\begin{pmatrix} \gamma_{11} & 0 & \gamma_{13} & \gamma_{14} \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$
$A_{3,1} \oplus A_1$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$q^1$
$A_{3,2} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^5$
$A_{3,3} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & 0 & 0 & 1 \end{pmatrix}$	$q^1, q^2$
$A_{3,4} \oplus A_1$	$\begin{pmatrix} 1 & \gamma_{12} & 0 & \gamma_{14} \\ \gamma_{12} & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^5$

TABLE 2 (Continued)

Lie Algebra	(Possible) Reducible Form of $\gamma_{\alpha\beta}$	Functionally Independent Invariants
$A_{3,5}^\alpha \oplus A_1$	$\begin{pmatrix} 1 & \gamma_{12} & 0 & \gamma_{14} \\ \gamma_{12} & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^5$
$A_{3,6} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^5$
$A_{3,7}^\alpha \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^3, q^5, q^6$
$A_{3,8} \oplus A_1$ and $A_{3,9} \oplus A_1$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & \gamma_{34} \\ \gamma_{14} & \gamma_{24} & \gamma_{34} & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$
$A_{4,1}$	$\begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 & 0 \\ \gamma_{12} & \gamma_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$q^1, q^3, q^5$
$A_{4,2}^\alpha$	$\begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{12} & 1 & \gamma_{23} & 0 \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & 0 & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$

TABLE 2 (Continued)

Lie Algebra	(Possible) Reducible Form of $\gamma_{\alpha\beta}$	Functionally Independent Invariants
$A_{4,2}^1$	$\begin{pmatrix} 1 & 0 & \gamma_{13} & 0 \\ 0 & 1 & \gamma_{23} & 0 \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & 0 & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5$
$A_{4,3}$	$\begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{12} & 1 & 0 & 0 \\ \gamma_{13} & 0 & \gamma_{33} & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^5$
$A_{4,4}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} & 0 \\ 0 & \gamma_{23} & \gamma_{33} & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^5$
$A_{4,5}^{\alpha,\beta}$	$\begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{12} & 1 & \gamma_{23} & 0 \\ \gamma_{13} & \gamma_{23} & 1 & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^5$
$A_{4,5}^{\alpha,\alpha}$	$\begin{pmatrix} 1 & \gamma_{12} & 0 & 0 \\ \gamma_{12} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1, q^2$
$A_{4,5}^{\alpha,1}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & 0 \\ 0 & \gamma_{23} & 1 & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1, q^2$
$A_{4,5}^{1,1}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1$

TABLE 2 (Continued)

Lie Algebra	(Possible) Reducible Form of $\gamma_{\alpha\beta}$	Functionally Independent Invariants
$A_{4,6}^{\alpha,\beta}$	$\begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{12} & 1 & 0 & 0 \\ \gamma_{13} & 0 & \gamma_{33} & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{pmatrix}$	$q^1, q^3, q^4, q^5$
$A_{4,7}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5$
$A_{4,8}$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{23} & 1 & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5$
$A_{4,9}^{\beta}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5$
$A_{4,9}^1$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \gamma_{34} \\ 0 & 0 & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3$
$A_{4,9}^0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5$
$A_{4,10}$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5$

TABLE 2 (Continued)

Lie Algebra	(Possible) Reducible Form of $\gamma_{\alpha\beta}$	Functionally Independent Invariants
$A_{4,11}^\alpha$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^3, q^4, q^5, q^6$
$A_{4,12}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} & \gamma_{33} \\ 0 & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$

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